# Computation of Radially Symmetric Shocked Flows*

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A practical method for computing solutions to partial differential equations with radial symmetry is presented. It consists of the following steps: 1. Consider the problem in Cartesian coordinates but store the data only at lattice points of an appropriate lower dimensional subspace. 2. Use interpolation and the known symmetries of the solution, determine the data at those mesh points of the subspace which are neighbors of the points described in step 1. 3. Compute at the lattice points of step 1 using a numerical scheme appropriate to Cartesian coordinates.

To demonstrate the method, an implosion calculation is performed and the results compared with other methods.

## I. INTRODUCTION

Many problems in hydrodynamics have cylindrical or spherical symmetry. When the equations of motion are written in the corresponding coordinates the following features appear:

  a.   There are singular terms proportional to $1/r$.

  b.   The momentum equation is not in *conservation form*.

These features cause difficulties mainly around the origin which are usually overcome by some ad hoc device such as extrapolation near the origin, etc. [see Payne 4].

In this paper we propose a method which is not ad hoc but uses the *conservation form* in the original Cartesian coordinates [1]. We present a numerical example to show that the method is not only logically appealing but it also works. It consists essentially of the following steps:

    1.   Consider the problem in Cartesian coordinates but store the data only at lattice points of a lower dimensional subspace—the $x$ axis in problems of spherical symmetry and the $x$–$z$ plane for problems which are invariant under rotation about the $z$ axis.

2.   Use interpolation and the known symmetries of the solution, determine the data at those meshpoints of the subspace which are neighbors of the points described in step 1.

3.   Compute at the lattice points of step 1 using a numerical scheme appropriate to Cartesian coordinates.

We shall call our method the Cartesian Method. Its advantage is that there is already an extensive theory of numerical computation in Cartesian space [1, 2, 3], and, furthermore, the singularity usually introduced at the origin along with the polar coordinate system is removed.

An interesting property of the Cartesian method is that flow problems with tangential (i.e., non-radial) momentum components can be computed as easily as those with only radial momentum. Consider a rotating cylinder, for example. Letting $m(r, \theta)$, $n(r, \theta)$ denote the components of momentum for the point $(r, \theta)$, we have by radial symmetry that

$$m(r, \theta) = m(r, 0) \cos \theta - n(r, 0) \sin \theta,$$

$$n(r, 0) = m(r, 0) \sin \theta + n(r, 0) \cos \theta.$$

Later it will be seen that these equations are used for part of step 2 of the Cartesian method and that for the problem considered, the tangential momentum component $n(r, 0)$ vanishes.

The Cartesian method will be illustrated by solving the following converging shock problem solved previously by Payne [4]:

Payne's problem is stated in cylindrical coordinates $(r, \theta)$. The initial conditions are:

$$\begin{array}{llll} p = 1, & & p = 4, & \\ \rho = 1, & \text{for} \ \ r \leqslant 1, & \rho = 4, & \text{for} \ \ r > 1, \\ m = 0, & & m = 0. & \end{array}$$

Section 2 of this report describes the differential equations used, and Section 3 describes the difference equations and interpolation and some aspects of the organization of the computer program.

In Section 5, the results obtained using the Cartesian method are compared with those obtained by Payne.

## II. DIFFERENTIAL EQUATIONS

Following step 1 we consider the problem in Cartesian coordinates even though the problem is naturally stated in one fewer dimension in polar coordinates. The

differential equations used are the conservation form of the equations of time dependent two-dimensional compressible fluid dynamics.

$$U_t + F_x + G_y = 0, \tag{1}$$

where

$$U = \begin{bmatrix} \rho \\ m \\ n \\ e \end{bmatrix}, \qquad F(U) = \begin{bmatrix} m \\ m^2/\rho + p \\ mn/\rho \\ (e+p)\, m/\rho \end{bmatrix}, \qquad G(U) = \begin{bmatrix} n \\ mn/\rho \\ n^2/\rho + p \\ (e+p)\, n/\rho \end{bmatrix},$$

in which

$x, y$  are cartesian coordinates

$t$      is time

$\rho$   is mass per unit volume

$u$      is the horizontal velocity component

$v$      is the vertical velocity component

$m$      is $\rho \cdot u$

$n$      is $\rho \cdot v$

$e$      is total energy per unit volume

$p$      is given by $e = p/(\gamma - 1) + \rho(u^2 + v^2)/2$

$\gamma$ is the ratio of specific heats (we are assuming a gamma-law gas)

The initial conditions are of the form $U(x, y, 0) = \phi(x^2 + y^2)$. That is, the initial flow quantities depend only on the distance from the origin.

Although the Eqs. (1) used for these examples are those of compressible fluid dynamics, other systems of partial differential equations—for example, the equations of magnetohydrodynamics—can be put into the same form. Because jumps in the flow quantities representing shocks or contact discontinuities, are allowed to exist in the flow, the solution sought is the *weak* solution to (1). That is, see [2] a piecewise continuous function $U(x, y, t)$ is sought which is a solution to the integral equation $\iiint (w_t U - w_x F - w_y G)\, dx\, dy\, dt + \iint w(x, y, 0)\, \phi(x^2 + y^2)\, dx\, dy = 0$ for all smooth test vectors $w$ which vanish for $|x| + t$ large enough.

In this approach, the numerical methods lead to shocks and contact discontinuities which are not boundaries but interior regions of rapid variation of the flow quantities. While this conservation law approach is not absolutely necessary for the systematic computation of symmetric flows using the method presented here, its use will allow us to refer to the literature [1–3, 5, 6] for experience on questions of accuracy, stability and consistency.

In summary, the equations used are in *rectangular* coordinates even though the problems and solutions are assumed to have radial symmetry and thus a "natural" mathematical representation in polar coordinates. This is the main idea of the numerical method which will be discussed in the next section.

## III. Difference Equations and Computational Procedure

In this section, the difference scheme used for the computed solution of Eq. (1) is described although some of the schemes for solving (1), e.g., [1–3, 6], might be used with suitable modifications. The overall difference procedure consists of two main parts; a standard difference scheme for two space dimensions such as [1–3, 5, 6] and an interpolation. After these two parts are described in detail for the problems solved, the consistency and truncation error of the overall difference procedure will be discussed. Because no stability proof is given here, it is still necessary to be careful not to transform a standard difference scheme which is stable into an overall scheme which is not.

The standard scheme is a variation of the two step $L - W$ described in [5].

The first step of the two-step scheme is

$$U^{n+\frac{1}{2}}_{j+\frac{1}{2},\ k+\frac{1}{2}} = \frac{1}{4}\left(U^n_{jk} + U^n_{j+1\ k} + U^n_{j\ k+1} + U^n_{j+1\ k+1}\right)$$

$$- \frac{\varDelta t}{2\varDelta x}\left(F^n_{j+1\ k+\frac{1}{2}} - F^n_{j\ k+\frac{1}{2}}\right) - \frac{\varDelta t}{2\varDelta y}\left(G^n_{j+\frac{1}{2}\ k+1} - G^n_{j+\frac{1}{2}\ k}\right).$$

The second step of the scheme is

$$U^{n+1}_{jk} = U^n_{jk} - \frac{\varDelta t}{\varDelta x}\left(\bar{F}^{n+\frac{1}{2}}_{j+\frac{1}{2}\ k} - \bar{F}^{n+\frac{1}{2}}_{j-\frac{1}{2}\ k}\right) - \frac{\varDelta t}{\varDelta y}\left(\bar{G}^{n+\frac{1}{2}}_{j\ k+\frac{1}{2}} - \bar{G}^{n+\frac{1}{2}}_{j\ k-\frac{1}{2}}\right),$$

in which

$$U^n_{jk} = \begin{bmatrix} \rho^n_{jk} \\ m^n_{jk} \\ n^n_{jk} \\ e^n_{jk} \end{bmatrix}, \quad F^n_{jk} = \begin{bmatrix} m^n_{jk} \\ (m^n_{jk})^2/\rho^n_{jk} + p^n_{jk} \\ m^n_{jk} \cdot n^n_{jk}/\rho^n_{jk} \\ (e^n_{jk} + p^n_{jk}) \cdot m^n_{jk}/\rho^n_{jk} \end{bmatrix}, \quad G^n_{jk} = \begin{bmatrix} n^n_{jk} \\ m^n_{jk} \cdot n^n_{jk}/\rho^n_{jk} \\ (n^n_{jk})^2/\rho^n_{jk} + p^n_{jk} \\ (e^n_{jk} + p^n_{jk}) n^n_{jk}/\rho^n_{jk} \end{bmatrix}$$

and

$$\bar{F}^{n+\frac{1}{2}}_{j+\frac{1}{2}\ k} = \left(F^{n+\frac{1}{2}}_{j+\frac{1}{2}\ k+\frac{1}{2}} + F^{n+\frac{1}{2}}_{j+\frac{1}{2}\ k-\frac{1}{2}}\right)/2;$$

$$\bar{G}^{n+\frac{1}{2}}_{j\ k+\frac{1}{2}} = \left(G^{n+\frac{1}{2}}_{j+\frac{1}{2}\ k+\frac{1}{2}} + G^{n+\frac{1}{2}}_{j-\frac{1}{2}\ k+\frac{1}{2}}\right)/2.$$

This is a nine-point scheme because nine points of the mesh in $x - y$ space at step $n$ are used to obtain numerical values at a point at step $n + 1$. This scheme can be used only for a point at which flow quantities are known for all its eight neighbors. In [5] it is shown that this scheme is consistent and has truncation error $O(\Delta x^3)$ where $\Delta x$ is the mesh spacing.
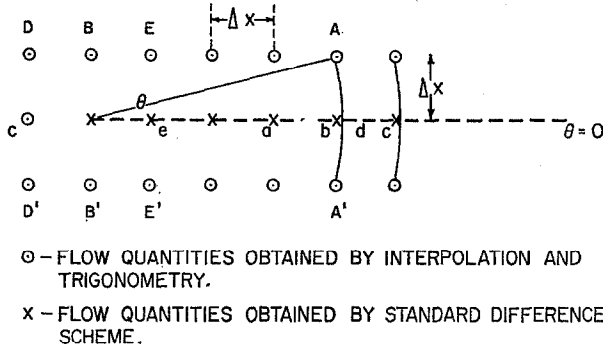


⊙ – FLOW QUANTITIES OBTAINED BY INTERPOLATION AND
     TRIGONOMETRY.

✕ – FLOW QUANTITIES OBTAINED BY STANDARD DIFFERENCE
     SCHEME.

FIG. 1.   Geometry of "Cartesian Method."

Consider Fig. 1. The horizontal line is the $x$ axis. At points on it, the vertical component of momentum $n$, is equal to 0. To obtain the flow quantities at a point such as $A$, an interpolation is performed using the points $a$, $b$, $c$ to find the flow quantities at $d$, which is at the intersection of the $x$ axis and the circle with center at the origin that goes through the point $A$. Let $\theta$ be the angle which the ray from the origin to $A$ makes with the $x$ axis. Let $(\rho_d, m_a, n_a, e_a)$ be the flow quantities (obtained by interpolation) at the point $d$. Then, assuming radial symmetry for the solution, the corresponding flow quantities at the point $A$ are $(\rho, m \cos \theta, -m \sin \theta, e)$.

Using this same method, the points near the origin at $B$, $C$, $B'$ get flow quantities assigned to them which are appropriately related to those at $e$. Values at the point $E$ are obtained in the same way as those at $A$ while those at $D$ and $D'$ are obtained from those at $E$ by symmetry.

The quantities $\sin \theta$ and the $\cos \theta$ are independent of the flow values and thus need to be computed only once in the calculation rather than at every step. In fact, because the distances from the origin to point $d$ and point $b$ in Fig. 1 need to be computed anyway for the interpolation coefficients, the computation of $\sin \theta$ and $\cos \theta$ reduces to a division each.

At point $d$, a Langrangean quadratic polynomial interpolation which has error $O(\Delta x^3)$ is used. Let the distances from the origin to the points $a$, $b$, $c$, $d$, be denoted by $r_a, r_b, r_c, r_d$. Then the interpolation formula for obtaining the approximate

value $\hat{f}$ of a function $f$ at $d$ assuming that the values at $a$, $b$, $c$ are known is given by

$$\hat{f}(d) = a_1 f(a) + a_2 f(b) + a_3 f(c),$$

where

$$a_1 = (r_d - r_b)(r_d - r_c)/(2\Delta x^2),$$
$$a_2 = (r_d - r_a)(r_d - r_c)/(-\Delta x^2),$$
$$a_3 = (r_d - r_a)(r_d - r_b)/(2\Delta x^2),$$

and $\Delta x$ is the spacing between the points $a$, $b$, $c$.

In [5] it is shown that if quantities are known at step $n$ with error $0(\Delta^3)$, then the application of the "standard" part of this difference method gives quantities which have $0(\Delta x^3)$ error at the $n + 1$st step also. Since the interpolation and symmetry part of this scheme doesn't introduce errors greater than $0(\Delta^3)$ compared with those produced by the "standard" scheme, and since the "standard" part of the scheme produces errors of order at most $0(\Delta^3)$, provided initial data was of order $\phi(\Delta^3)$, the overall error committed is $0(\Delta^3)$. This proves that the truncation error is $0(\Delta^3)$.

*Proof of Consistency.*

Define the operators $N(U)$ and $N_\Delta(U)$ by the equations

$$N(U) = U_t + F_x + G_y,$$

and

$$N_\Delta(U) = (U_{jk}^{n+1} - U_{jk}^n)/\Delta t + (\bar{F}_{j+\frac{1}{2}\,k}^{n+\frac{1}{2}} - \bar{F}_{j-\frac{1}{2}\,k}^{n+\frac{1}{2}})/\Delta x + (\bar{G}_{j\,k+\frac{1}{2}}^{n+\frac{1}{2}} - \bar{G}_{j\,k-\frac{1}{2}}^{n+\frac{1}{2}})/\Delta y.$$

$N_\Delta(U)$ is the difference operator corresponding to the "standard" part of the scheme. Define

$$N_\Delta'(U) = (U_{jk}'^{n+1} - U_{jk}'^n)/\Delta t + (\bar{F}_{j+\frac{1}{2}\,k}'^{n+\frac{1}{2}} - \bar{F}_{j-\frac{1}{2}\,k}'^{n+\frac{1}{2}})/\Delta x + (\bar{G}_{j\,k+\frac{1}{2}}'^{n+\frac{1}{2}} - \bar{G}_{j\,k-\frac{1}{2}}'^{n+\frac{1}{2}})/\Delta y,$$

where primes denote quantities after the interpolation and reflection. Now, to show consistency, we need to show that $| N_\Delta(U) - N(U)| \to 0$ as $\Delta t, \Delta t \to 0$. From the consistency of $N_\Delta(U)$ with $N(U)$, we have $| N_\Delta(U) - N(U)| = 0(\Delta) \to 0$ as $\Delta \to 0$. Thus we need only show that $| N_\Delta(U) - N_\Delta'(U) | \to 0$ as $\Delta \to 0$. But now observe that $N_\Delta'(U) = N_\Delta(U + 0(\Delta^3)) = N_\Delta U + 0(\Delta^3)$, since $N_\Delta$ is Lipschitz continuous (away from discontinuities). Thus $| N_\Delta'(U) - N_\Delta(U)| = | 0(\Delta^3)| \to 0$ as $\Delta \to 0$. This proof can be extended to any scheme in which the error of the interpolation goes to zero with the mesh spacing.

## IV. Computing Procedure

The following gives the computing procedure in a step-by-step form:

One: Compute and store initial data for the flow.

Two: Compute and store interpolation coefficients and sin $\theta$ and cos $\theta$ for each point of type A in Fig. 1.

Three: Perform interpolations and transformations. Store resulting data at points of type $A$, $B$, $C$, $D$, $E$ and $A'$, $B'$, $D'$, $E'$ of Fig. 1.

Four: Perform standard difference scheme iteration for points on $x$ axis except $C$.

Five: Obtain necessary output and return to Three.

If storage space in the computer were crucial while a small amount of machine time and programmer effort were not, then steps two, three and four could be merged at each point of type a. The distances from the origin to the point of type A could be computed efficiently by using a Newton–Raphson iteration with a first guess taken as the distance computed at the previous point. Using such a conceptually more complex approach would save much of the additional storage otherwise used by the Cartesian method.

## V. Results of Numerical Methods

A cylindrical shock problem which was originally solved by Payne [4] was used for illustration and comparison. The initial conditions, stated in polar coordinates $(r, \theta)$ are

$$\begin{array}{llll} \rho = 1, & & \rho = 4, & \\ p = 1, & \text{for } r \leqslant 1, & p = 4, & \text{for } r > 1, \\ m = 0, & & m = 0. & \end{array}$$

These conditions give a shock with initial strength 1.93, a contact discontinuity, and an expansion fan. The simulation was carried out for the scheme reported here and compared to the results of Payne. There were unexplained differences; so Payne's scheme was redone. Although the present graphs, obtained by using Payne's difference equations, are similar in structure to his original graphs, they differ mainly in the time $t$ that the shock takes to reach the origin of symmetry. These discrepancies occur perhaps because the time step is taken in a different way.

In the more recent simulation, $\Delta t/\Delta r$ seems to be slightly smaller than that which Payne reported he took. The effect would be to increase diffusion, and thus the shock would (and does) reach the center sooner.

## A. Comparison of the Graphs

We will consider only the differences between the pairs of graphs displayed here. The major difference is that the shock reaching the center leads to
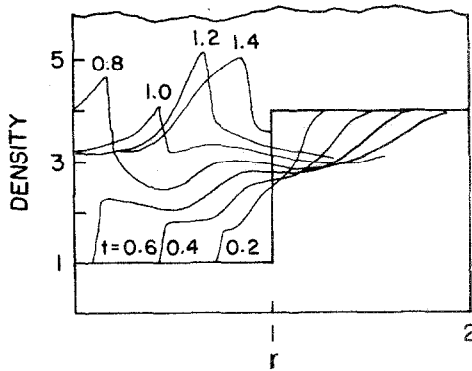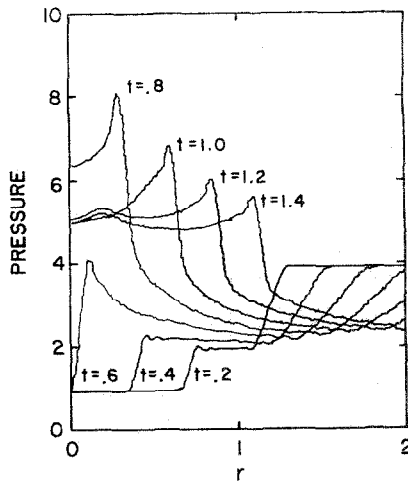


FIG. 2.   Density computed by Payne (Redrawn).



FIG. 3.   Pressure computed by Payne (Redrawn).
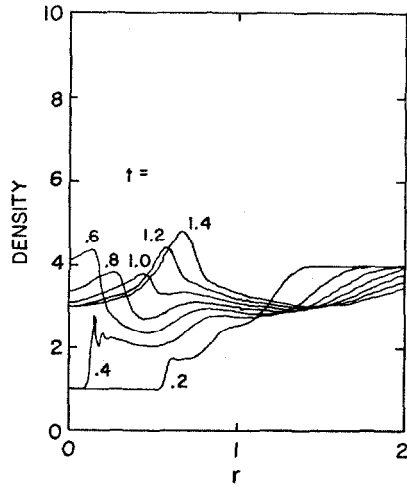
FIG. 4.   Density computed by "Cartesian Method" based on Lax differencing.
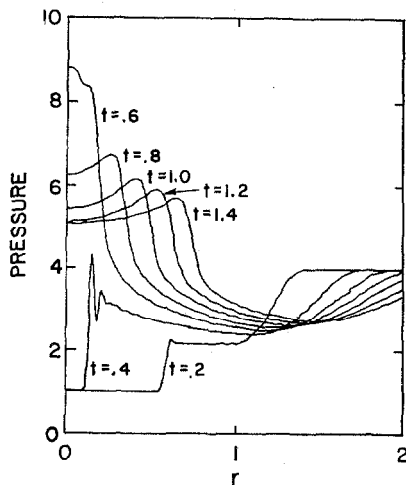
FIG. 5.   Pressure computed by "Cartesian Method" based on Lax differencing.
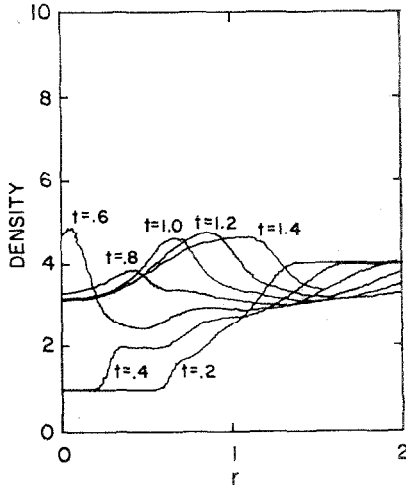
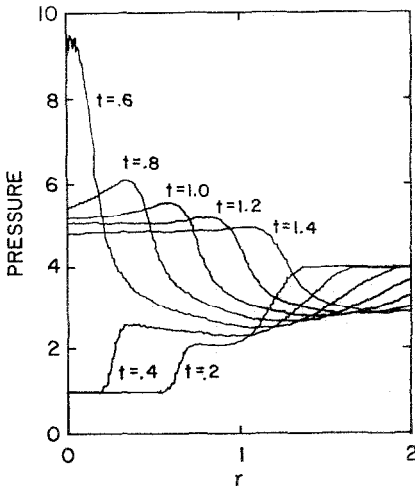FIG. 6.   Density by replication of Payne's method.



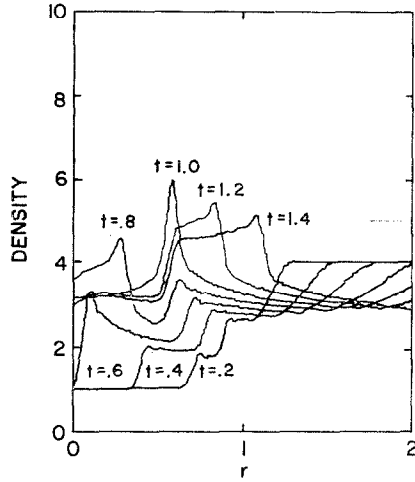FIG. 7.   Pressure by replication of Payne's method.

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FIG. 8.  Density computed by "Cartesian Method" based on Lax–Wendroff differencing.
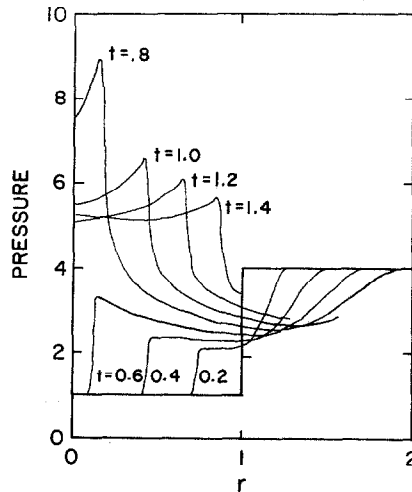


FIG. 9.  Pressure computed by "Cartesian Method" based on Lax–Wendroff differencing.

nothing spectacular in the Cartesian method. In fact, the shock diverging from the center looks very much like the shock converging to it, while by Payne's method, the diverging shock has large-amplitude short-wave oscillations which later die out. More spectacular perhaps is the very clear contact discontinuity which appears and persists throughout the Cartesian computation and which becomes blurred almost immediately in the computation of Payne. In the density

graph of the Cartesian computation (Fig. 8)—for example, at $t = .40$—there are three clearly defined steps. In the corresponding graphs of Payne, (Fig. 2 and Fig. 6), the second steps are much less obvious.

Comparing the pressures curves, the curves at 1.0, 1.2, 1.4 from the "Cartesian method" (Fig. 9) have an additional small maximum to the left of the shock. Because the Cartesian method is second-order accurate and Payne's method is first-order so, it is likely that the relative maximum is missed because of the relative inaccuracy.

## B. Details of the Computation

For the Cartesian method the parameters of the computation are $\Delta r = .01$, $\Delta t = .002$ and since the problem has been reduced to one in rectangular coordinates, all the knowledge which is available for such problems can be used. In particular, the artificial viscosity reported in [5] can be used to avoid the numerical instabilities which otherwise occur. The coefficient of artificial viscosity was taken to be 4.

For the simulation which used Payne's method, $\Delta r = .01$ and there was no appended artificial viscosity term. In an attempt to redo Payne's computation, $\Delta t$ was taken by estimating the shock speed $U$ by the first-order formula proven in [7]:

$$U = \tfrac{1}{2}(u_0 + c_0 + u + c), \tag{2}$$

assuming that this term would be the largest near the shock. Then $\Delta t/\Delta r = $ const $U^{-1}$ was used. Experiments with constant lead to a value of .4 for stability in this computation. Payne used a different method for finding $U$ and still using $\Delta t/\Delta r = $ const$/U$, he used const $= .75$. Because (2) is an underestimate for $U$, both computations gave the same average value of $\Delta t/\Delta r$.

The foregoing discussion demonstrates the "Cartesian method" based on the Lax–Wendroff difference scheme. The computation is not quite comparable to Payne's scheme, because it is second-order accurate. In order to have a more direct basis for comparison, the Cartesian (two-dimensional) Lax difference scheme was also used as a basis for a "Cartesian method" first-order accurate computation. The interpolation part was taken to be linear because higher order accuracy would have been wasted. The results are illustrated by the unretouched graphs of Figs. 4 and 5. The maximum pressure at the origin obtained for this case was 88 vs 30 for the replication of Payne's method. The correct value is infinite.

The computations were done on a finite mesh of points. In order to obtain advanced values at the outermost point ($r = 2$), linear extrapolation from advanced quantities at $r - \Delta r$ and $r - 2\Delta r$ were used at the end of each step.

CONCLUSION

These results clearly demonstrate the power of the "Cartesian method" for computing flows with radial symmetry. This method gives very good resolution of shocks and contact discontinuities and allows the full machinery of the theory of numerical analysis which has been developed for rectangular coordinates to be applied to such problems.

REFERENCES

1. P. D. LAX, *Comm. Pure Appl. Math.* 7 (1954), 159.
2. P. D. LAX AND B. WENDROFF, *Comm. Pure Appl. Math.* 13 (1960), 217.
3. R. D. RICHTMYER AND K. W. MORTON, "Difference Methods for Initial Value Problems," 2nd. Ed., Interscience, New York, 1967.
4. R. B. PAYNE, *J. of Fluid Mechanics* 2 (1957), 185.
5. A. LAPIDUS, *J. of Computational Physics* 2 (1967), 154.
6. R. D. RICHTMYER, "A Survey of Difference Methods for Non-Steady Gas Dynamics," National Center for Atmospheric Research, Technical Note 63-2, Boulder, Colo. 1963.
7. R. COURANT AND K. O. FRIEDRICHS, "Supersonic Flow and Shock Waves," p. 159, Interscience, New York, 1948.